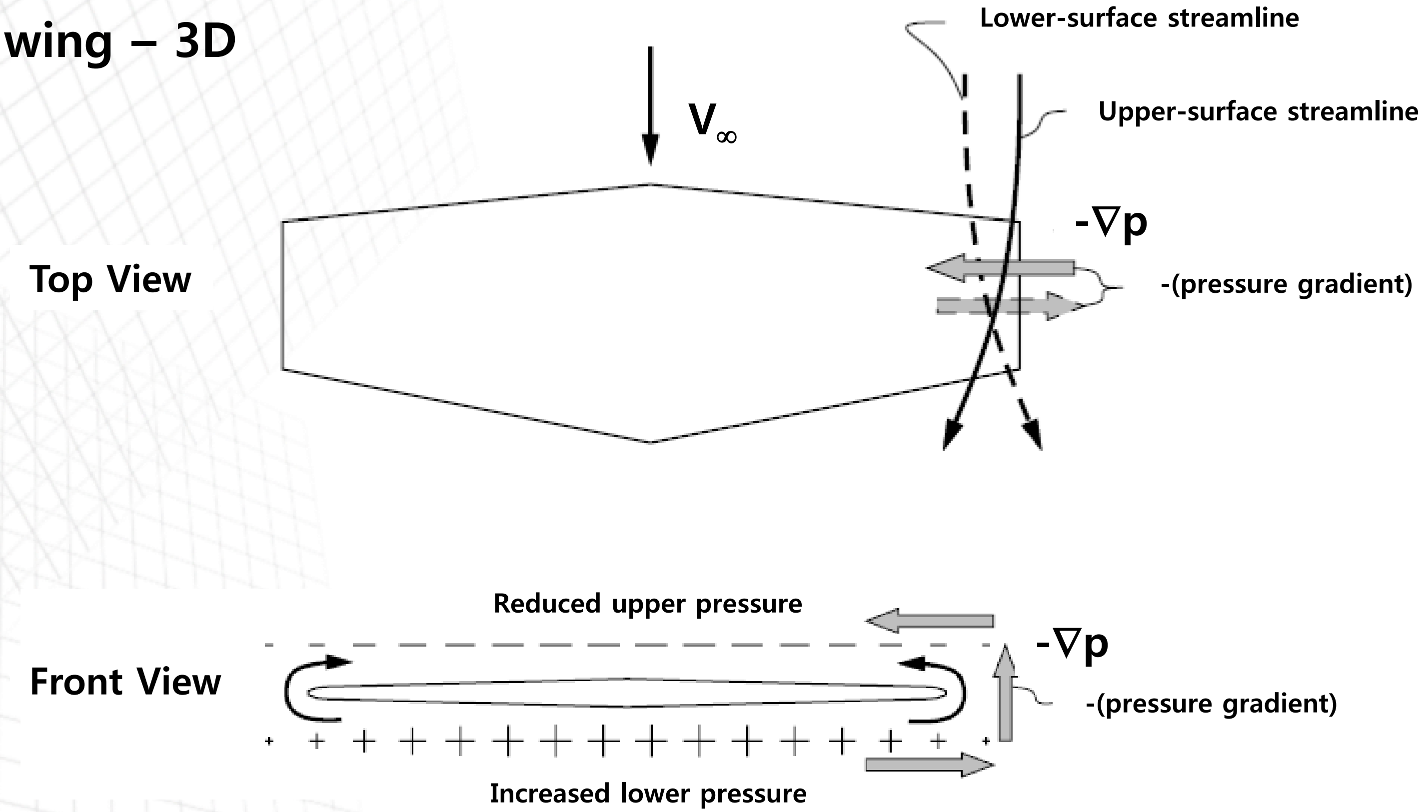
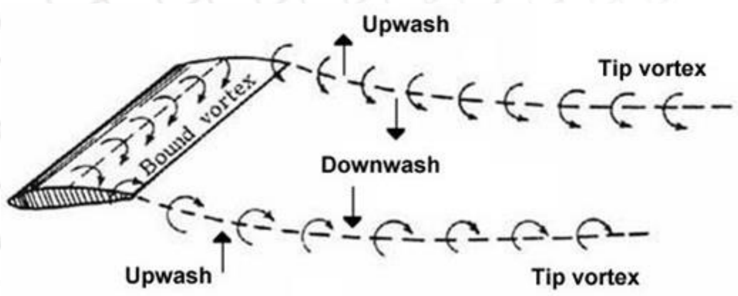
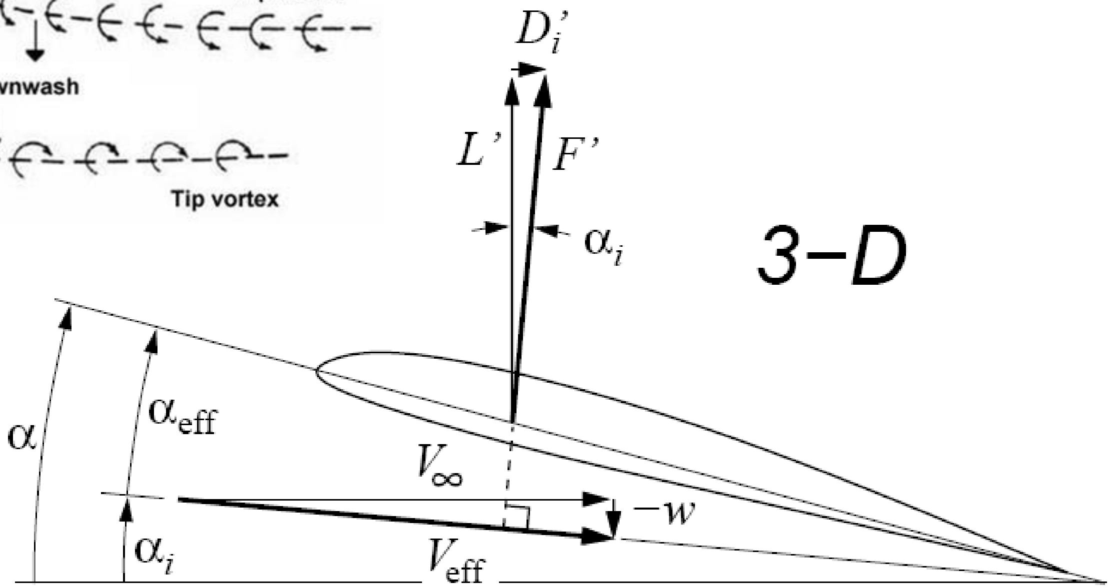
< 5.1 Downwash and Induced Drag >

* Infinite wing – airfoil (2D)
* Finite wing – 3D



< 5.1 Downwash and Induced Drag >





α : Geometric AOA

α_i : Induced AOA

α_{eff} : Effective AOA = a - a_i

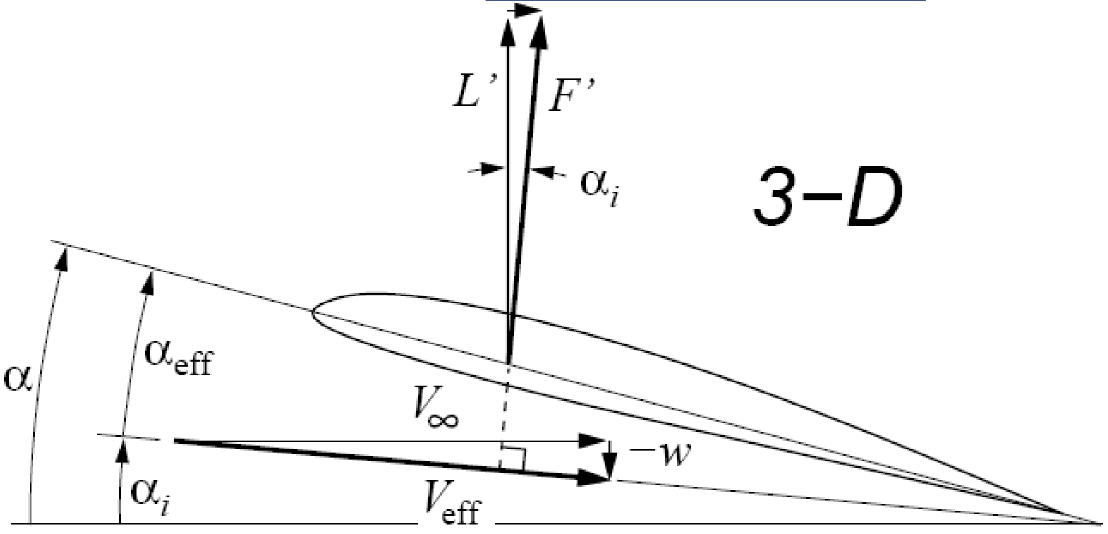
D_i : Induced drag

w : Downwash

D'Alembert's paradox is not valid anymore.

< 5.1 Downwash and Induced Drag >

D_i : Induced drag



* Total drag = profile drag + induced drag

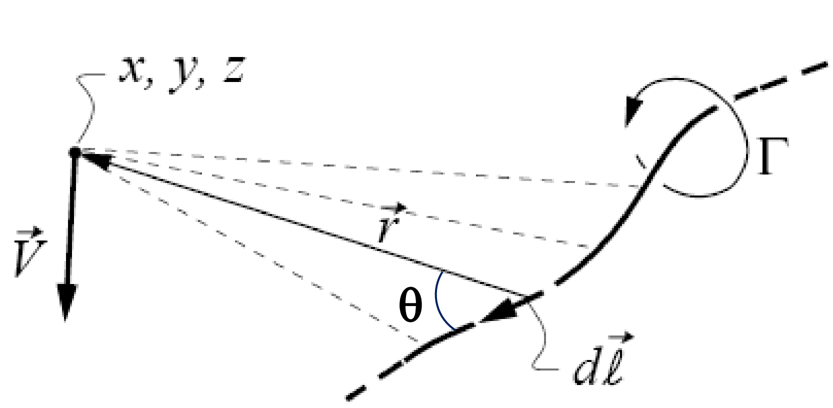
= pressure drag + skin friction drag + induced drag

$$D_{total} = D_p + D_f + D_i$$

< 5.2 The Biot-Savart Law and Helmholtz's Theorems>

❖ The Biot-Savart Law

* vortex filament



$$d\vec{V} = \frac{\Gamma}{4\pi}\frac{d\vec{l}\times\vec{r}}{|r|^3}$$

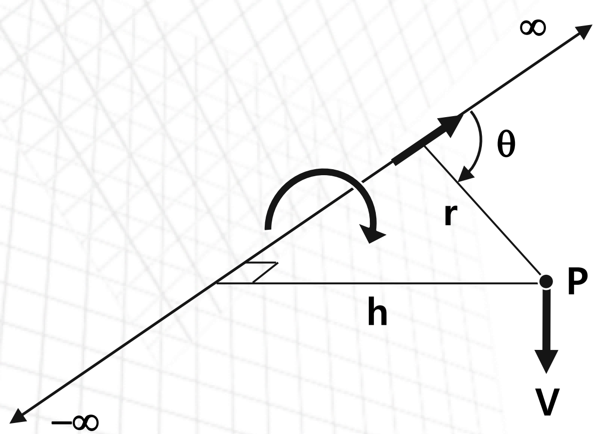
$$\vec{V} = \int_{-\infty}^{\infty}\frac{\Gamma}{4\pi}\frac{d\vec{l}\times\vec{r}}{|r|^3}$$

$$V = |\vec{V}| = \frac{\Gamma}{4\pi}\int_{-\infty}^{\infty}\frac{\sin\theta}{r^2}dl$$

< 5.2 The Biot-Savart Law and Helmholtz's Theorems>

❖ The Biot-Savart Law

* An infinite straight vortex filament



$$r = \frac{h}{\sin\theta},\ l = \frac{h}{\tan\theta},\ dl = -\frac{h}{\sin^2\theta}d\theta$$

$$\Rightarrow\ V = \frac{\Gamma}{4\pi}\int_{-\infty}^{\infty}\frac{\sin\theta}{r^2}dl$$

$$= \frac{\Gamma}{4\pi}\int_{\pi}^{0}\frac{\sin\theta}{r^2}\left(-\frac{h}{\sin^2\theta}\right)d\theta$$

$$= \frac{\Gamma}{4\pi}\int_{\pi}^{0}\frac{\sin^3\theta}{h^2}\left(-\frac{h}{\sin^2\theta}\right)d\theta$$

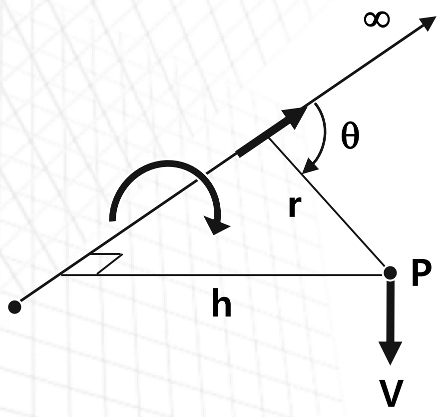
$$= -\frac{\Gamma}{4\pi h}\int_{\pi}^{0}\sin\theta\, d\theta = -\frac{\Gamma}{4\pi h}[-\cos\theta]_{\pi}^{0}$$

$$= \frac{\Gamma}{2\pi h}$$

< 5.2 The Biot-Savart Law and Helmholtz's Theorems>

❖ The Biot-Savart Law

* An Semi-infinite straight vortex filament



$$V = \frac{\Gamma}{4\pi h}$$

< 5.2 The Biot-Savart Law and Helmholtz's Theorems>

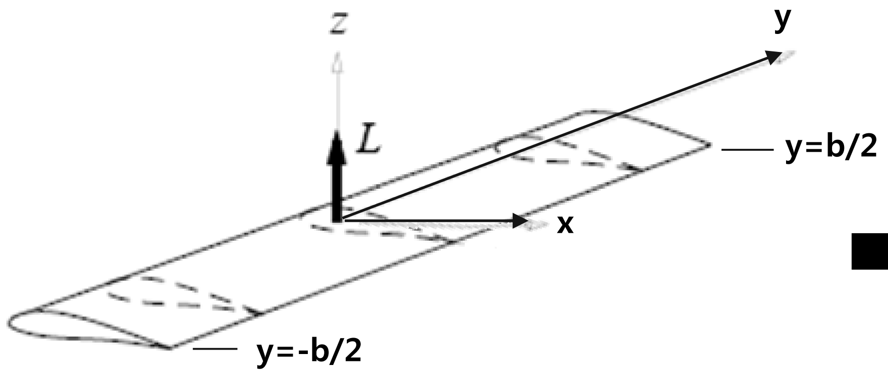
❖ Helmholtz's theorems

* Vortex strength is constant along the filament

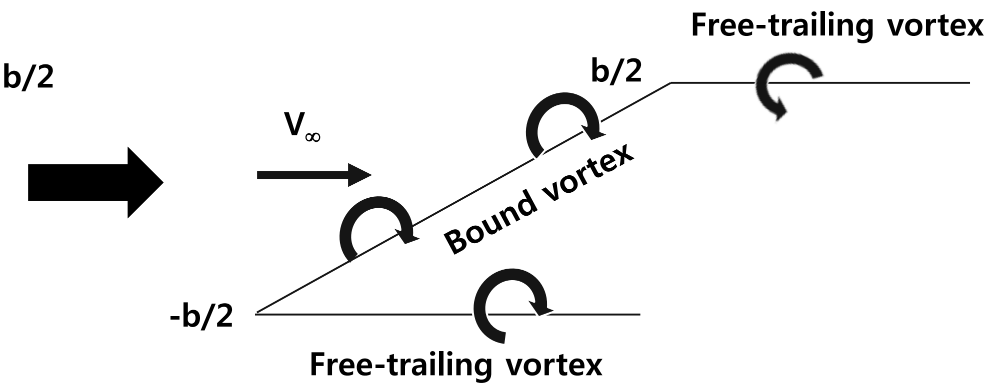
* Vortices are forever (never end)

* Vortices move with the flow

< 5.3 Prandtl's Classical Lifting-Line Theory>



Finite wing



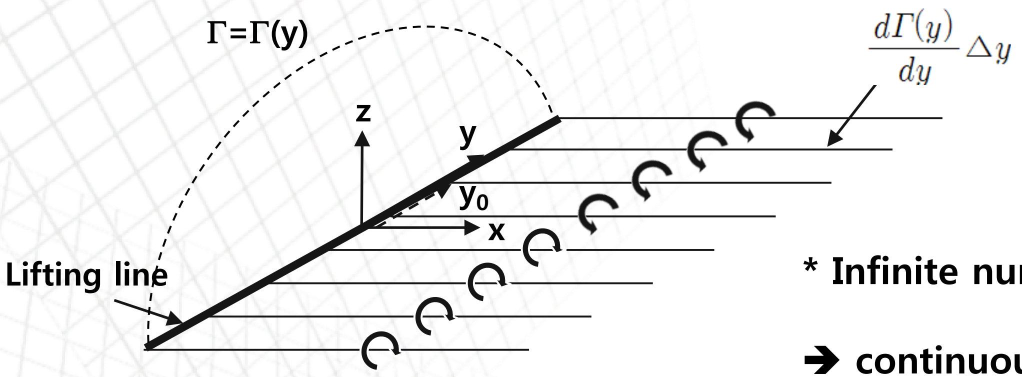
Horseshoe vortex

* The velocity at any point y along the bound vortex induced by the trailing semi-infinte vortices

$$\Rightarrow \quad w(y) = -\frac{\Gamma}{4\pi(\frac{b}{2}+y)} - \frac{\Gamma}{4\pi(\frac{b}{2}-y)} = -\frac{\Gamma}{4\pi}\frac{b}{(\frac{b}{2})^2 - y^2}$$

< 5.3 Prandtl's Classical Lifting-Line Theory>

* Superimposition of a large number of horseshoe vortices



* Infinite number of horseshoe vortices

➔ continuous distribution of $\Gamma(y)$ along the lifting line

* The velocity *dw* at y_0 induced by the entire semi-infinite trailing vortex located at y

➔ $$dw = -\frac{(d\Gamma/dy)dy}{4\pi(y_0 - y)}$$

* Then the total velocity *w* induced at y_0 by the entire trailing vortex sheet

➔ $$\boxed{w(y_0) = -\frac{1}{4\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{(d\Gamma/dy)dy}{y_0 - y}} \quad \text{............... (1)}$$

< 5.3 Prandtl's Classical Lifting-Line Theory>

* The induced angle of attack α_i at the local airfoil section of a finite wing of the arbitrary spanwise station y_0

➔ $$\alpha_i(y_0) = \tan^{-1}\left(\frac{-w(y_0)}{V_\infty}\right)$$

* If $\left(\frac{w(y_0)}{V_\infty}\right) \ll 1$ ➔ $\alpha_i = -\frac{w(y_0)}{V_\infty}$ .... (2)

* Substituting eq. (1) into eq. (2)

➔ $$\boxed{\alpha_i(y_0) = \frac{1}{4\pi V_\infty}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{(d\Gamma/dy)dy}{y_0 - y}dy}$$ .... (3)

< 5.3 Prandtl's Classical Lifting-Line Theory>

* The lift coefficient for the airfoil section at $y=y_0$

➔ $$C_l = a_0\left[\alpha_{eff}(y_0) - \alpha_{L=0}\right] = 2\pi\left[\alpha_{eff}(y_0) - \alpha_{L=0}\right] \quad \text{............... (4)}$$

* From the definition of lift coefficient and the Kotta-Joukowski theorem

➔ $$L' = \frac{1}{2}\rho_\infty V_\infty^2 c(y_0) C_l = \rho_\infty V_\infty \Gamma(y_0)$$

$$\therefore C_l = \frac{2\Gamma(y_0)}{V_\infty c(y_0)} \quad \text{............... (5)}$$

* Substituting eq. (5) into eq. (4)

➔ $$\alpha_{eff} = \frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \alpha_{L=0} \quad \text{............... (6)}$$

< 5.3 Prandtl's Classical Lifting-Line Theory>

* Substitution of eq.(3) and (6) into the equation for α_{eff} , $\alpha_{eff} = \alpha - \alpha_i$

➔ $$\boxed{\underbrace{\alpha(y_0)}_{\text{Geometric AOA}} = \underbrace{\frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \alpha_{L=0}(y_0)}_{\text{Effective AOA}} + \underbrace{\frac{1}{4\pi V_\infty}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{(d\Gamma/dy)dy}{y_0 - y}}_{\text{Induced AOA}}}$$

➔ The fundamental equation of Prandtl's lifting line theory

* In the above equation, the only unknown is Γ

< 5.3 Prandtl's Classical Lifting-Line Theory>

* By the solution $\Gamma=\Gamma(y_0)$, following aerodynamic characteristics of finite wing are obtained.

1. The lift distribution

$$L'(y_0) = \rho_\infty V_\infty \Gamma(y_0)$$ from the Kutta-Joukowski theorem

2. The total lift

$$L = \int_{-\frac{b}{2}}^{\frac{b}{2}} L'(y)dy = \rho_\infty V_\infty \int_{-\frac{b}{2}}^{\frac{b}{2}} \Gamma(y)dy \quad \text{............... (*)}$$

$$C_L = \frac{L}{q_\infty S} = \frac{2}{V_\infty S}\int_{-\frac{b}{2}}^{\frac{b}{2}} \Gamma(y)dy$$

3. The induced drag

$D_i' = L_i' \sin\alpha_i$: The induced drag per unit span

$D_i' = L_i' \alpha_i$ ← if α_i is small

$$D_i = \int_{-\frac{b}{2}}^{\frac{b}{2}} L'(y)\alpha_i(y)dy = \rho_\infty V_\infty \int_{-\frac{b}{2}}^{\frac{b}{2}} \Gamma(y)\alpha_i(y)dy$$: The total induced drag

$$C_{D,i} = \frac{D_i}{q_\infty S} = \frac{2}{V_\infty S}\int_{-\frac{b}{2}}^{\frac{b}{2}} \Gamma(y)\alpha_i(y)dy$$: The induced drag coefficient .... (**)